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Bending waves on inviscid columnar vortices

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Bending waves, perturbation modes leading to deflections of the vortex centreline, are considered for an infinitely long straight vortex embedded in an irrotational flow of unlimited extent. We first establish the general form of the dispersion relation for long waves on columnar vortices with arbitrary distributions of axial and azimuthal vorticity by a singular perturbation analysis of the Howard–Gupta equation. The asymptotic results are shown to compare favourably with numerical solutions of the Howard–Gupta equation for wavelengths comparable to the vortex core radius and longer. Dispersion relations are then found numerically for specific models of vortex core structures observed experimentally; here no restrictions are placed on wavelength. The linear dispersion relation has an infinite number of branches, falling into two families; one with infinite phase speed at zero wavenumber (which we call 'fast' waves), the other with zero phase speed ('slow' waves). In the long-wave limit, slow waves have zero group velocity, while the fast waves may have finite non-zero group speeds that depend on the form of the velocity profiles on the axis of rotation. Weakly nonlinear waves are described under most circumstances by the nonlinear Schrödinger equation. Solitons are possible in certain 'windows' of wavenumbers of the carrier waves. An example has already been presented by Leibovich & Ma (1983), who compute solitons and soliton windows on a fast-wave branch for a vortex with a particular core structure. Experimental data of Maxworthy, Hopfinger & Redekopp (1985) reveal solitons which appear to be associated with the slow branch, and these are computed for velocity profiles fitting their data. The nonlinear Schrödinger equation is shown to fail for long waves, and to be replaced by a nonlinear integro-differential equation.

1. Introduction

Concentrated vortices act as waveguides capable of supporting dispersive waves of various kinds. That this is so has been known for a long time, waves on a Rankine vortex being treated by Kelvin in 1880. Experimental observations of propagating waves are also of long standing, although with few exceptions, such as Pritchard's (1970) studies of axisymmetric solitons, the observational evidence is not systematic or well-documented. Recent experiments by Hopfinger, Browand & Gagne (1982) and by Maxworthy, Hopfinger & Redekopp (1985 hereinafter referred to as MHR) together provide systematic data elucidating the behaviour of non-axisymmetric waves on vortices. Furthermore, the former paper suggests that these waves may be important as mediators between turbulence caused by local agitation (such as exists in the ocean near the air–sea interface) and the bulk of a rotating fluid in a quasi-geostrophic and ordered balance. The waves appear to transport energy and momentum from the turbulent agitation zone into the essentially turbulence-free

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bulk: by their occasional breakdown, turbulence is injected deep in the interior of the bulk fluid.

Despite the long history of the subject, available theoretical results are scattered and confined to a few special cases with discontinuous velocity or vorticity distributions, such as the treatment by Kelvin cited above and an extension by Krishnamoorthy (1966), but allowing for a constant axial velocity confined to the core, and by Pocklington (1895) for waves on a hollow vortex.

What is most needed, in many cases, is information about the propagation of waves having wavelength long compared to the diameter of the vortex core. In this paper, we derive the dispersion relations and modal forms (eigenfunctions) for long non-axisymmetric inviscid waves on columnar vortices embedded in an irrotational flow, where the vortices are allowed to have essentially arbitrary continuous velocity profiles. A columnar vortex is one which, in the most general case, has zero radial velocity but non-zero azimuthal and axial velocities depending on radial distance from an axis of symmetry. We look for waves for which the perturbation velocity vector, in the infinitesimal amplitude limit, takes the form $\boldsymbol{u} = \hat{\boldsymbol{u}}(r)\exp[\mathrm{i}(kz+m\theta-\omega t)]$, where m is an integer azimuthal wavenumber, k is an axial wavenumber, and ω is the frequency. We work out what we believe to be the complete solution for the case of long bending waves, which are those permitting fluid particles initially on the symmetry axis to be displaced from it. There are an infinite number of branches of the dispersion relation $\omega = \omega(k)$ for such waves. Viewed in a coordinate system moving with the axial velocity at large distances from the axis, one branch has, at $k = 0$, zero frequency, phase, and group velocity. This we call the slow wave. All other branches have non-zero frequency, infinite phase velocity, and finite group velocity (which depends upon the local properties of the axial and angular velocities of the vortex near the axis of rotation) as $k \rightarrow 0$. These we call the fast waves.

The dispersion relation on the slow branch for small wavenumbers has been given, essentially as we find it here, by Moore & Saffman (1972). Their derivation of it involves fitting a Biot–Savart velocity field, with cutoff determined by a local steady flow found by Widnall, Bliss & Zalay (1971), to the motion of a helicoidal vortex filament. This procedure is *ad hoc*, while the present treatment is rigorous (at least according to accepted singular perturbation theory) and is therefore, in our view, preferable.

Waves in the fast branch are, in fact, neutrally propagating centre-modes, whose structure for unstable concentrated vortices has been explained by Stewartson & Brown (1985) and they may be found by an analytical procedure similar to that used in their problem. Furthermore, analysis of the centre-mode behaviour is not much influenced by the azimuthal wavenumber, m. By contrast, our method of finding the slow wave structure depends on $|m| = 1$, so it applies only to bending waves: it is conceivable that the slow branch exists only in the case $|m| = 1$, but we as yet have no evidence that this is so.

Our treatment of the slow branch for $|m| = 1$ is to be found in §3 and that for the fast branches for all $|m| \geqslant 1$ in §4. In both cases, we find that the asymptotic results agree very well with corresponding information from direct numerical integration of the relevant eigenvalue problem. Our comparisons are done for flows selected from the four-parameter family of vortices with axial velocity, W, and azimuthal velocity, V, given by

$$W(r) = W_0 \exp(-\alpha_1 r^2), \qquad (V)$$

$$V(r) = \frac{\Gamma}{2\pi r}(1-\exp(-\alpha_2 r^2)).$$

Similar forms have been used frequently to fit experimental data. For example, Maxworthy *et al.* (1985) find that the choices $W_0 = 0.4$, $\alpha_1 = 0.54$, $\alpha_2 = 1.28$, $\Gamma/2\pi = 1.39$ fit their data.

Leibovich & Ma (1983) considered the propagation of bending waves of finite amplitude, assuming weak nonlinearity, and attempted to use this solution to fix the undetermined constant in the local induction approximation (LIA) to the Biot–Savart formula (this approximation is conveniently found in Batchelor 1967). These waves are slowly modulated about a centre carrier wavenumber by an amplitude governed by the cubically nonlinear Schrödinger (NLS) equation. MHR concluded that the solutions found may be related to experimental observations for short waves, but bore little relation to most of the waves they observed. The waves considered by Leibovich & Ma were determined by numerical solutions on the primary fast branch of the vortex (V) with $W_0 = 0$, $\alpha_2 = 1$. Soliton solutions to the NLS equation were sought and found to be possible only in a band ('soliton window') of carrier (axial) wavenumbers in the approximate interval $0.68 < k < 1$. The data of MHR clearly show that most waves observed were on the slow branch. We therefore reconsider the problem of Leibovich & Ma in §5, and look for soliton windows on the slow branch. This is carried out numerically for a wide range of wavenumbers (not just for long waves) for two examples of (V), $W_0 = 0$, $\alpha_2 = 1$, $\Gamma/2\pi = 1$, the case considered by Leibovich & Ma and called 'flow A' in this paper, and $W_0 = 0.4$, $\Gamma/2\pi = 1.39$, $\alpha_1 = 0.54$, $\alpha_2 = 1.28$, of MHR called 'flow B' here. Now soliton windows are found centred on $|k| = 0$ with short-wave cutoffs, and in other windows comprising intervals in k not including $k = 0$.

The nonlinear Schrödinger equation fails to provide a valid description of wave motion for very long carrier waves on the slow branch. The failure may be traced to a singularity in the dispersion relation, which formally leads to infinite values of one of the NLS coefficients. This is corrected in §6, where the NLS is found to be replaced by a nonlinear integrodifferential equation. Thus the modulation envelope is no longer locally determined, but is controlled in part by events at distant portions of the vortex, as would be predicted by the Biot–Savart formula. This linkage of remote portions of the vortex may be attributed physically to the presence of an outer irrotational flow of infinite extent.

The modal amplitudes, $\hat{\boldsymbol{u}}(r)$, for slow bending waves disturb the entire core, but those on the fast branch are concentrated near the axis of rotation as the wavenumber decreases. As a consequence, the presence of long fast waves may be difficult to detect, while long slow waves should be readily detectable. We also find features of the geometry and motion of phase fronts, which take the form of twisted ribbons, that distinguish between long fast and slow bending waves. These features are explained in §7, which also contains additional concluding remarks.

2. Problem formulation

We postulate the existence of a stable columnar vortex embedded in an incompressible and inviscid fluid of infinite axial extent. In a cylindrical (r, θ, z)-coordinate system this motion is described by the velocity vector

$$\boldsymbol{U} = (0, V(r), W(r)). \tag{1}$$

While not required for all of our subsequent work, we assume that the radial extent of the fluid is infinite, that the vorticity associated with (1) decays exponentially fast as $r \rightarrow \infty$, and that the circulation about the axis of rotation approaches a non-zero constant in this limit. The vorticity is then concentrated within a characteristic distance, a_0, of the axis, which we take as the unit of length. A characteristic speed V_0 of the flow (1) is selected as unit for velocities, and a_0/V_0 is taken as unit for time. Henceforth, all variables are taken to be dimensionless with the scales indicated. We adopt that coordinate system in which $\lim_{r\to\infty} W(r) = 0$, and we assume that $\lim_{r\to\infty} rV = \Gamma/2\pi$ in our dimensionless variables.

Infinitesimal perturbations to (1) correspond to isolated neutral waves; there are by hypothesis no contiguous unstable modes. The velocity vector is

$$\boldsymbol{v} = \boldsymbol{U}(r) + \epsilon\boldsymbol{u}(r,\theta,z,t;\epsilon), \tag{2}$$

were ϵ is a small amplitude parameter. It is assumed that the perturbation velocity vector can be expanded in a power series in ϵ

$$\boldsymbol{u}(r,\theta,z,t;\epsilon) = (u,v,w) = \boldsymbol{u}_1 + \epsilon\boldsymbol{u}_2 + \epsilon^2\boldsymbol{u}_2 + \dots, \tag{3}$$

with a similar expansion for pressure.

The Euler equations require $\boldsymbol{u}(r,\theta,z,t;\epsilon)$ to satisfy the equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{L}\boldsymbol{u} + \boldsymbol{\nabla} p = \epsilon \boldsymbol{N}, \tag{4a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \equiv \left(\frac{1}{r}\frac{\partial}{\partial r} ru\right) + \left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right) + \frac{\partial w}{\partial z} = 0, \tag{4b}$$

where $\mathbf{L}$ is the matrix differential operator

$$\left(\frac{V}{r}\frac{\partial}{\partial\theta} + W\frac{\partial}{\partial z}\right)\boldsymbol{I} + \begin{pmatrix} 0 & -2\Omega & 0 \\ \mathrm{D}_* V & 0 & 0 \\ \mathrm{D}W & 0 & 0 \end{pmatrix}, \tag{5}$$

with Ω, D and D_* defined to be

$$\Omega \equiv \frac{V}{r}, \quad \mathrm{D} \equiv \frac{\partial}{\partial r}, \quad \mathrm{D}_* \equiv \mathrm{D} + \frac{1}{r}, \tag{6}$$

$\boldsymbol{I}$ is the identity matrix and $\boldsymbol{N} = (N_1, N_2, N_3)$ is the bilinear vector with components

$$N_1 = -\left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + w\frac{\partial u}{\partial z} - \frac{1}{r}v^2\right), \tag{7a}$$

$$N_2 = -\left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right), \tag{7b}$$

$$N_3 = -\left(u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + w\frac{\partial w}{\partial z}\right). \tag{7c}$$

The linearized problem, (4) with $\epsilon = 0$, admits solutions in the form of waves

$$\left.\begin{aligned} \boldsymbol{u} &= A\hat{\boldsymbol{u}}(r)\exp(\mathrm{i}\psi) + \text{c.c.} \\ p &= A\hat{p}(r)\exp(\mathrm{i}\psi) + \text{c.c.}, \end{aligned}\right\} \tag{8}$$

where c.c. represents the complex conjugates of the terms displayed and ψ is the phase function

$$\psi \equiv kz + m\theta - \omega t, \tag{9}$$

which depends on the axial wavenumber k (real), the azimuthal wavenumber m (integer), and the frequency ω. The wavenumber vector,

$$\boldsymbol{\kappa} = \boldsymbol{\nabla}\psi = (0, m/r, k),$$

is normal to surfaces of constant phase: between any two constant values of the radius these surfaces take the form of twisted ribbons.

For fixed k and m, the linearized problem is an eigenvalue problem with eigenvalue $\omega(k,m)$ and corresponding eigenvector $\boldsymbol{u}(\boldsymbol{r})$. This linear problem will be treated in detail in §§3 and 4, with emphasis on bending waves, which are those with $m = \pm 1$, and on the long wave limit $k \rightarrow 0$.

Boundary conditions on the solution to (4) are taken to be

$$\lim_{r\rightarrow\infty} \boldsymbol{u} = 0,$$

and single-valuedness of $\boldsymbol{u}$ and the vorticity vector on the axis $r = 0$. The latter requirements imply that, if a solenoidal vector $\boldsymbol{u}$ differentiable at $r = 0$ is expanded in the Fourier series

$$\boldsymbol{u} = \sum_{m=-\infty}^{\infty} \boldsymbol{u}^{(m)}(r,z,t)\,\mathrm{e}^{\mathrm{i}m\theta},$$

then

$$w^{(m)}(0,z,t) = 0 \quad (m \neq 0), \tag{10a}$$

$$\frac{\partial w^{(m)}}{\partial r}(0,z,t) = 0 \quad (\text{for all } m), \tag{10b}$$

$$u^{(m)}(0,z,t) = v^{(m)}(0,z,t) = 0 \quad (|m| \neq 1), \tag{10c}$$

$$\frac{\partial}{\partial r} u^{(m)}(0,z,t) = \frac{\partial}{\partial r} v^{(m)}(0,z,t) = 0 \quad (|m| = 1). \tag{10d}$$

Bending waves permit fluid particles on the axis of rotation to be deflected off the axis: from (10c) this is possible only when $|m| = 1$.

We identify $\boldsymbol{u}_1$ with the solution (8) of the linearized problem for a particular constant wavenumber pair $k = k_0$, $m = m_0$ and seek higher-order terms in (3) by substitution into (4). The series (3) will not be well-ordered for $\epsilon t = O(1)$ unless secular terms are suppressed. This can be done by allowing A to vary slowly with time or space or both. Here we permit modulation in both space and time: this may be accomplished, as is well known, by introducing a slow space scale

$$Z = \epsilon(z - C_{\mathrm{g}}t), \tag{11}$$

and a slow timescale

$$\tau = \epsilon^2 t. \tag{12}$$

With $\boldsymbol{u}_1$ and p_1 given in the form (8), the problems for $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are of the form

$$\frac{\partial \boldsymbol{u}_2}{\partial t} + \mathsf{L}\boldsymbol{u}_2 + \boldsymbol{\nabla}p_2 = (|A|^2 + A^2\,\mathrm{e}^{2\mathrm{i}\psi})\,\boldsymbol{R}_1(\hat{\boldsymbol{u}}_1,\hat{\boldsymbol{u}}_1) + \mathrm{e}^{\mathrm{i}\psi}\,A_Z[C_{\mathrm{g}}\,\hat{\boldsymbol{u}}_1 - \boldsymbol{C}_1\,\hat{\boldsymbol{u}}_1] + \text{c.c.} \tag{13}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_2 = 0,$$

$$\frac{\partial \boldsymbol{u}_3}{\partial t} + \mathsf{L}\boldsymbol{u}_3 + \boldsymbol{\nabla}p_2 = \left(|A|^2 A\boldsymbol{R}_2(\hat{\boldsymbol{u}}_1,\hat{\boldsymbol{u}}_2) + \mathrm{i}\boldsymbol{C}_2\,\hat{\boldsymbol{u}}_1\,\frac{\partial^2 A}{\partial Z^2} - \hat{\boldsymbol{u}}_1\,\frac{\partial A}{\partial \tau}\right)\mathrm{e}^{\mathrm{i}\psi} + A^3\,\mathrm{e}^{3\mathrm{i}\psi}\,\boldsymbol{R}_3(\hat{\boldsymbol{u}}_1,\hat{\boldsymbol{u}}_2) + \text{c.c.} \tag{14}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_3 = 0.$$

In these equations, the $\boldsymbol{R}_j$, $j = 1,2,3$ are bilinear vector-valued functions of their arguments, and $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are real and constant matrices. Particular solutions to these inhomogeneous equations free from secular growth exist if the right-hand sides are orthogonal to the adjoint linear problem. This requirement determines the linear group velocity C_g and requires A to satisfy the cubic Schrödinger equation

$$\mathrm{i}\frac{\partial A}{\partial \tau}+\hat{\mu}\frac{\partial^2 A}{\partial Z^2}+\hat{\nu}A|A|^2=0, \tag{15}$$

with $\hat{\mu}$ and $\hat{\nu}$ determined from (14) as integrals of the scalar product of the adjoint solution vector with $\boldsymbol{C}_2\boldsymbol{u}_1$ and $\boldsymbol{R}_2$ respectively. The numbers C_g and $\hat{\mu}$ may be found, alternatively, from the dispersion relation, with $C_g=\omega'(k_0)$ and $\hat{\mu}=\frac{1}{2}\omega''(k_0)$, since the two methods are equivalent. As the carrier wavenumber, k_0, tends to zero, the cubic Schrödinger equation fails and must be replaced. This situation is considered in §6.

The procedure that we have followed is to reduce the problem for $\hat{\boldsymbol{u}}_1$ to a single homogeneous second-order ordinary differential equation for the radial component $\hat{u}_1(r)$. Similarly, we look for the various particular solutions for $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ by first reducing the problem to a second-order inhomogeneous differential equation. Particular solutions for the resonant cases have the same ordinary differential operator that acts on $\hat{u}_1(r)$: non-resonant particular solutions have different differential operators.

The problem for $\hat{u}_1$ may be put in the following form:

$$\mathsf{L}\hat{u}_1\equiv \mathrm{D}(S\mathrm{D}_*\hat{u}_1)-(1+\gamma^{-1}a+\gamma^{-2}b)\,\hat{u}_1=0, \tag{16a}$$

with boundary conditions

$$\left.\begin{aligned}\hat{u}_1(0)&=0 \quad \text{if } |m|\neq 1,\\ \mathrm{D}\hat{u}_1(0)&=0 \quad \text{if } |m|=1,\\ \hat{u}_1&\rightarrow 0 \quad \text{as } r\rightarrow\infty.\end{aligned}\right\} \tag{16b}$$

where

$$\left.\begin{aligned}S&=r^2(m^2+k^2r^2)^{-1},\\ \gamma&=kW+m\Omega-\omega,\\ a&=r\mathrm{D}\{(m^2+k^2r^2)^{-1}[kr\mathrm{D}W+m\mathrm{D}_*V]\},\\ b&=-2k\Omega r^{-1}S[kr\mathrm{D}_*V-m\mathrm{D}W].\end{aligned}\right\} \tag{17}$$

Equation (16*a*) was derived first by Howard & Gupta (1962) and we refer to it as the Howard–Gupta equation, or HGE. The coefficients in (16*a*), a, b, γ, and S, may be given in an alternative way (Leibovich 1986) that can be helpful in making physical interpretations. These are

$$a=r^2\boldsymbol{\nabla}\cdot\{r^{-2}(\boldsymbol{\kappa}\times\boldsymbol{\zeta})/|\boldsymbol{\kappa}|^2\};\quad b=-2(\boldsymbol{\kappa}\cdot\boldsymbol{\Omega})(\boldsymbol{\kappa}\cdot\boldsymbol{\zeta})/|\boldsymbol{\kappa}|^2;$$
$$\gamma=\boldsymbol{\kappa}\cdot\boldsymbol{U}(r)-\omega;\quad S=1/|\boldsymbol{\kappa}|^2$$

where $\boldsymbol{\kappa}$ is the wavenumber vector

$$\boldsymbol{\kappa}=\boldsymbol{\nabla}\psi,$$

$\boldsymbol{\zeta}=\operatorname{curl}\boldsymbol{U}$ is the unperturbed vorticity vector, and $\boldsymbol{\Omega}=(0,0,\Omega(r))$ is the angular velocity vector of a fluid particle about the axis of symmetry in the unperturbed motion.

If the interest is on bending waves, then we select m to be $+1$ or -1 in ψ. The symmetry $\omega(-k,-m)=-\omega(k,m)$ is required for the perturbations to be real, and may be verified directly from the form of the HGE when we restrict attention to real ω (neutrally stable modes). The symmetry implies that the complete dispersion relation may be constructed for given $|m|$ either if ω is known for positive m and all real k or if ω is known for positive k and both $m = |m|$ and $m = -|m|$.

For columnar vortices with arbitrarily prescribed velocity profiles $W(r)$ and $V(r)$, and arbitrary carrier wavenumbers k_0, one must solve the HGE numerically. Higher-order corrections require further numerical work to compute $C_g(k_0)$, and the Schrödinger equation coefficients $\hat{\mu}(k_0)$ and $\hat{\nu}(k_0)$. This has been carried out for two examples taken from the four parameter family of velocity profiles

$$\left.\begin{aligned} W(r) &= W_0 \exp(-\alpha_1 r^2), \\ V(r) &= (\Gamma/2\pi r)[1-\exp(-\alpha_2 r^2)]. \end{aligned}\right\} \tag{18}$$

The results of these computations are reported in §5, using a numerical method differing in inconsequential ways from that described in Leibovich & Ma (1983).

The case of long carrier waves, that is, $k_0 \to 0$, is amenable to analysis, and solutions of the HGE for bending waves for a general class of vortices can be found without reference to numerical approximations. This is described in §3 for slow waves. Fast waves are treated in §4, where it is found that solutions can be found for all m and so the treatment there is not specialized to bending waves only.

3. Long waves on the slow branch: results for general vortices

Numerical evidence indicates the existence of several branches of the dispersion relation for particular basic vortex flows. For one branch, both the frequency $\omega \to 0$ and the phase speed $c = \omega/k \to 0$ and group speed $C_g = \mathrm{d}\omega/\mathrm{d}k \to 0$ as $k \to 0$. For the remaining branches, $\omega \to m\Omega(0)$, $c \to \infty$ as $k \to 0$. A branch in the first category we call a slow wave, those in the second category we call fast waves. Slow waves are considered in this section, fast waves are treated in §4.

We seek solutions of the HGE (16a) subject to boundary conditions (16b) and asymptotically valid for $|k| \to 0$. The basic flow (1) is arbitrary in form, except for the restrictions stated in the beginning of §2. The treatment in this section is limited to bending waves, so $|m| = 1$. Both signs for m may be dealt with at once if we set

$$\beta = k/m, \quad \sigma = \omega/m; \tag{19}$$

then the HGE (16a) becomes

$$\mathrm{D}\left[\frac{r}{1+\beta^2 r^2}\,\mathrm{D}(r\chi)\right] - [m^2 + \bar{\gamma}^{-1}\bar{a} + \bar{b}\bar{\gamma}^{-2}]\chi = 0, \tag{20}$$

where the quantities $\bar{\gamma}$, $\bar{a}$, and $\bar{b}$ are found by the replacements $k \to \beta$, $m \to 1$, and $\omega \to \sigma$ in the expressions for γ, a, and b in (17) and the symbol χ is used for u_1.

The limit equation, with β set to zero in (20) is

$$r^2\chi'' + 3r\chi' - [(m^2-1) + (r/(\Omega-\sigma))\,\mathrm{d}/\mathrm{d}r(r\Omega' + 2\Omega)]\chi = 0. \tag{21}$$

If $m^2 = 1$, the general solution to (21) is

$$\chi = C_1[\Omega(r) - \sigma] + C_2(\Omega-\sigma)\int \frac{\mathrm{d}r}{r^3(\Omega-\sigma)^2}. \tag{22}$$

If $\Omega(0)$ is finite, as we assume, the coefficient of C_2 is singular at $r = 0$, and we therefore take $C_2 = 0$, and $C_1 = 1$. We next assume that $\Omega'(0) = 0$; this is a physical requirement for an axisymmetric vortex arising from a flow with viscosity.

Since $\Omega'(0) = 0$ and $\lim_{r\to\infty} \Omega(r) = 0$, $\chi(r)$ satisfies the boundary condition at the origin and $\chi \to -\sigma$ as $r \to \infty$. Therefore the boundary condition as $r \to \infty$ is satisfied if we choose $\sigma = 0$ and the eigenfunction as $\chi = \Omega(r)$. The form of the HGE then suggests a perturbation expansion in powers of β,

$$\chi(r;\beta) = \chi_0(r) + \beta\chi_1(r) + \beta^2\chi_2(r) + \dots,$$
$$\sigma(\beta) = \beta\sigma_1 + \beta^2\sigma_2 + \dots.$$

with $\chi_0(r) = \Omega(r)$.

This process fails at $O(\beta^2)$, it being impossible to satisfy the boundary condition at infinity. The failure can be traced to the existence of more than one lengthscale in the problem, one selected as unit for length and based on the core diameter and a second, a_0/k, based on the wavelength and tending to infinity as $k \to 0$. We therefore introduce an outer variable

$$\rho = |\beta|\, r, \tag{23}$$

and the HGE in the outer variable is

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\rho}{1+\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\rho\chi\right) - \chi = 0, \tag{24}$$

to all algebraic orders for $\rho = O(1)$ and $|\beta| \to 0$ under our assumptions of exponentially decaying vorticity as $r \to \infty$. The outer solution, satisfying the boundary condition at infinity, is

$$\chi = B(|\beta|)\frac{\mathrm{d}}{\mathrm{d}\rho}K_1(\rho), \tag{25}$$

where $K_1(\rho)$ is the modified Bessel function of order 1. The outer expansion is therefore developed as an asymptotic expansion of $B(|\beta|)$,

$$B(|\beta|) = B_1(|\beta|) + B_2(|\beta|) + \dots, \tag{26}$$

each with the same coefficient, $K_1'(\rho)$. As $\rho \to 0$,

$$\chi = B(|\beta|)\,K_1'(\rho) = \left[-\frac{1}{\rho^2} + \tfrac{1}{2}\ln(\tfrac{1}{2}\rho) + \tfrac{1}{4} + \tfrac{1}{2}\gamma_e + O(\rho)\right]B(|\beta|), \tag{27}$$

where γ_e is Euler's constant, 0.5772....

The inner solution, to lowest order, is

$$\chi = \Omega(r) \sim \frac{\Gamma}{2\pi r^2} \quad \text{as } r \to \infty, \tag{28}$$

and (27) and (28) match if we take

$$B_1 = -\frac{\beta^2\Gamma}{2\pi}.$$

The outer solution then forces a term $O(\beta^2 \ln|\beta|)$ in the inner eigenfunction and in σ. To this order in the inner solution no further terms in (26) need to be considered. We therefore take, in the inner region,

$$\left.\begin{aligned}\chi(r,\beta) &= \chi_0(r) + \beta\chi_1(r) + \beta^2\chi_2(r) + \beta^2\ln|\beta|\,\chi_3(r),\\ \sigma(\beta) &= \beta\sigma_1 + \beta^2\sigma_2 + \beta^2\ln|\beta|\,\sigma_3.\end{aligned}\right\} \tag{29}$$

If we let

$$G(r) = \frac{r^2\Omega'' + 3r\Omega'}{\Omega},$$

then

$$L_0\chi_0 \equiv r^2\chi_0'' + 3r\chi_0' - G(r)\,\chi_0 = 0, \tag{30a}$$

and the higher-order terms, χ_i, $i = 1,2,3$ satisfy

$$L_0 \chi_1 = \frac{R_1}{\Omega} \chi_0, \tag{30b}$$

$$L_0 \chi_2 = \frac{R_2}{\Omega} \chi_0 + [r^3(r\chi_0)']' + \frac{R_1}{\Omega} \chi_1, \tag{30c}$$

$$L_0 \chi_3 = \frac{R_3}{\Omega} \chi_0, \tag{30d}$$

where $$R_1 \equiv r^2 W'' + 3rW' - (W - \sigma_1)\, G(r),$$

$$R_2 \equiv -\frac{rW(rW')'}{\Omega} - r(r(r^2\Omega)')' - 2r(r^2\Omega)' + G(r)\left(\frac{W^2}{\Omega} + \sigma_2\right) - \frac{4rWW'}{\Omega},$$

and $$R_3 \equiv G(r)\,\sigma_3.$$

Boundary conditions at the axis require

$$\chi_i'(0) = 0. \tag{31}$$

Matching with the outer solution requires the following asymptotic behaviour as $r \to \infty$,

$$\chi_0(r) \sim \frac{\Gamma}{2\pi r^2}, \quad \chi_1(r) \to 0, \tag{32a, b}$$

$$\chi_2(r) \sim \frac{\Gamma}{4\pi}\{\ln r + \gamma_e + \tfrac{1}{2} - \ln 2\}, \tag{32c}$$

$$\chi_3(r) \sim -\frac{\Gamma}{4\pi}. \tag{32d}$$

The solution for χ_0 has already been found to be $\Omega(r)$. The particular solution to (30*b*) is

$$\chi_1 = -\sigma_1 + W(r), \tag{33}$$

which clearly satisfies the condition (31) on the axis, and can be made to satisfy (32*b*) by taking $\sigma_1 = 0$.

The solution of (30*c*) can be written as

$$\chi_2 = -\sigma_2 + \bar{\chi}_2(r), \tag{34}$$

where $\bar{\chi}_2$ is found by reduction of order to be

$$\bar{\chi}_2(r) = -\tfrac{1}{2}\Omega r^2 - \Omega \int_0^r \frac{d\xi}{\xi^3 \Omega^2(\xi)} \int_0^\xi [\eta^3 \Omega^2(\eta) + 2\eta^2 WW']\, d\eta, \tag{35}$$

The function χ_2 satisfies the axis conditions. Making use of the assumption that $\Omega \sim \Gamma/2\pi r^2$ as $r \to \infty$, we find that the inner integral in (35) behaves asymptotically like

$$\left(\frac{\Gamma}{2\pi}\right)^2 [\ln \xi + K]: \tag{36}$$

where K is the constant defined by

$$\left(\frac{\Gamma}{2\pi}\right)^2 K = \int_0^1 \eta^3 \Omega^2(\eta)\, d\eta + \int_1^\infty \eta^3 \left[\Omega^2(\eta) - \frac{\Gamma^2}{4\pi^2 \eta^4}\right] d\eta - \int_0^\infty 2\eta W^2(\eta)\, d\eta. \tag{37}$$

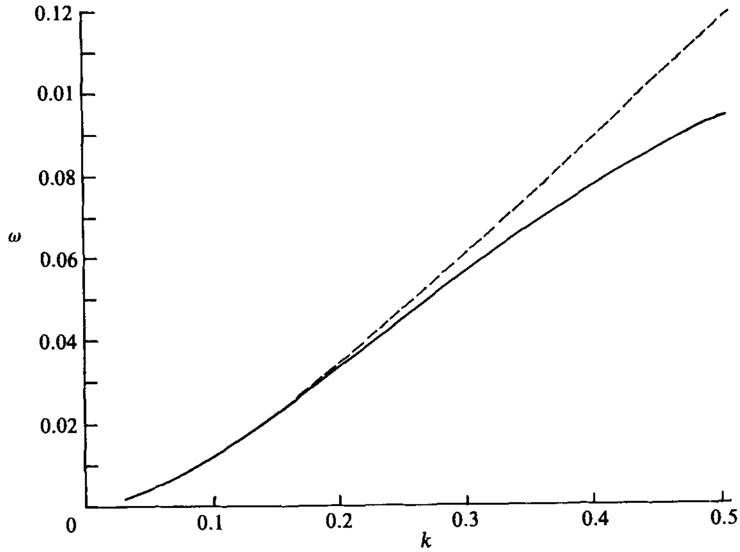

FIGURE 1. Slow branch dispersion relations for $m = -1$, for the flow A. The upper curve (dashed) is found by direct numerical computation of the eigenvalues of HGE. The lower curve (solid) is the asymptotic formula (41).

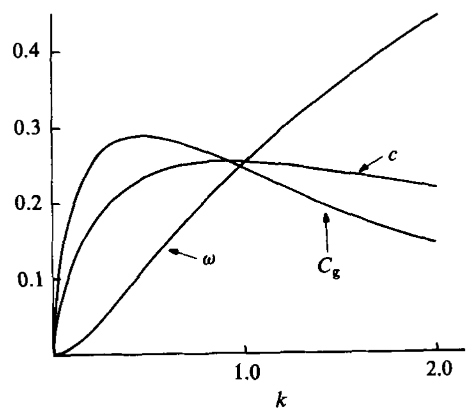

FIGURE 2. Frequency, phase and group speeds for the case of figure 1, as determined by direct numerical computation.

With the large r behaviour given by (36), χ_2 may be seen to have the following asymptotic behaviour:

$$\chi_2 \sim -\sigma_2 - \frac{\Gamma}{4\pi}\left[\ln r + K + \tfrac{1}{2}\right]. \tag{38}$$

Comparing (38) with (32c), we see

$$\sigma_2 = -\frac{\Gamma}{4\pi}\left[K + \ln 2 - \gamma_e\right]. \tag{39}$$

The solution to (30d) satisfying the axis condition is

$$\chi_3 = -\sigma_3,$$

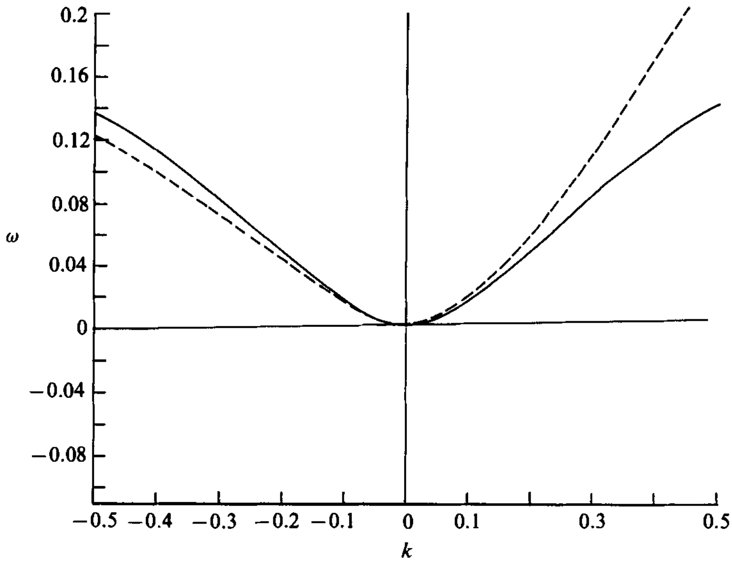

FIGURE 3. As in figure 1, but for flow B. The dashed curve is numerical, the solid curve is the asymptotic formula (41).

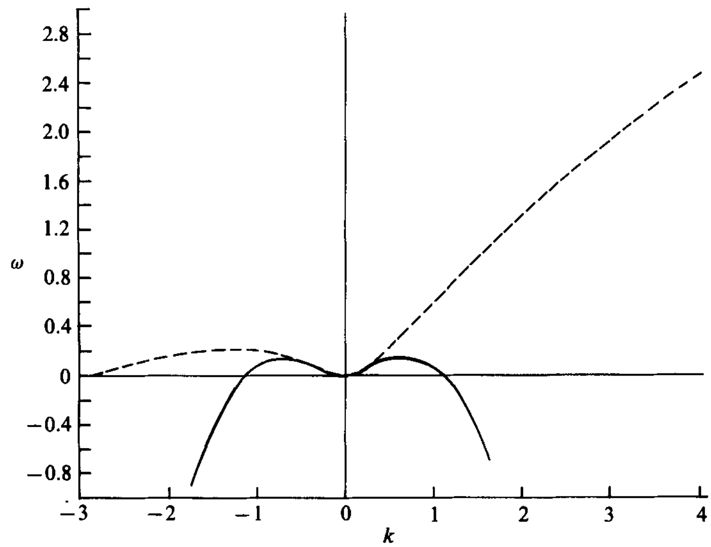

FIGURE 4. As in figure 3, but showing a larger wavenumber range.

and applying (32*d*) implies that

$$\sigma_3 = \frac{\Gamma}{4\pi}. \tag{40}$$

Thus, to $O(\beta^2)$ the dispersion relation is

$$\omega = m\sigma = -\frac{m\Gamma}{4\pi} k^2 \left[\ln\left(\frac{2}{|k|}\right) + K - \gamma_e \right], \tag{41}$$

where we have replaced $|\beta|$ by $|k|$.

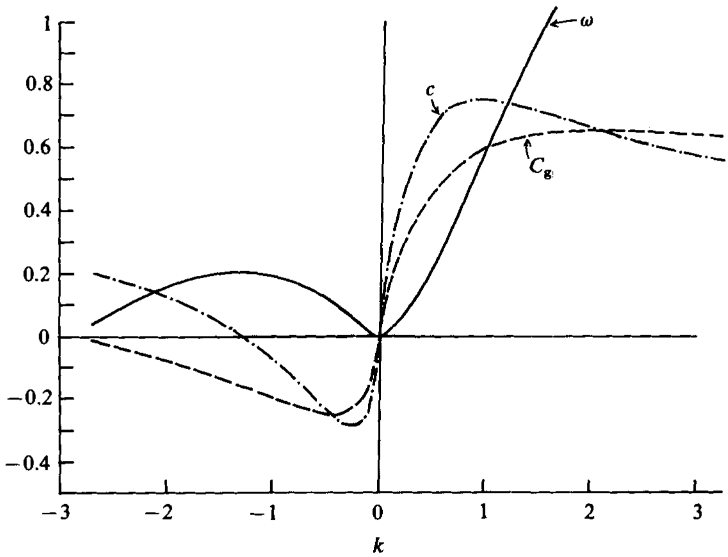

FIGURE 5. Like figure 2, but for flow B.

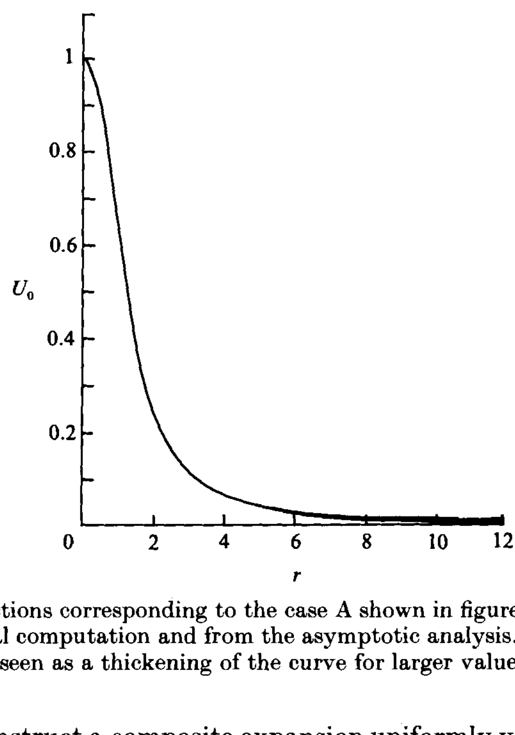

FIGURE 6. Eigenfunctions corresponding to the case A shown in figure 1, for $k = 0.10$, found from both direct numerical computation and from the asymptotic analysis. The results overlap, except for small differences seen as a thickening of the curve for larger values of r.

We may also construct a composite expansion uniformly valid in r from the results found above, with accuracy to $O[k^2 \ln(1/|k|)]$. This is

$$\chi_c(r) = \chi_0 + mk\chi_1(r) + k^2\chi_2 + k^2 \ln|k|\,\chi_3 - \frac{|k|\Gamma}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}K_1(|k|\,r)$$
$$-\frac{\Gamma}{2\pi r^2} + \frac{\Gamma}{4\pi}k^2[\ln(\tfrac{1}{2}|k|\,r) + \gamma_e + \tfrac{1}{2}],$$

or, written in terms of the vortex velocity profiles,

$$\chi_c(r) = \Omega(r)+mkW(r)-\frac{|k|\Gamma}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}K_1(|k|\,r)-\frac{\Gamma}{2\pi r^2}$$
$$+k^2\left\{\frac{\Gamma}{4\pi}[\ln r+K+\tfrac{1}{2}]-\frac{\Omega r^2}{2}-\Omega\int_0^r\frac{\mathrm{d}\xi}{\xi^3\Omega^2(\xi)}\int_0^\xi[\eta^3\Omega^2(\eta)+2\eta^2WW']\,\mathrm{d}\eta\right\}. \quad (42)$$

For reference, we note that the constant K for the family of profiles (18) is given by

$$K = \frac{1}{2}\left[\ln(\tfrac{1}{2}\alpha_2)+\gamma_e-\frac{4\pi^2\,W_0^2}{\alpha_1\,\Gamma^2}\right]. \quad (43)$$

The formula (41) for the dispersion relation has been given by Moore & Saffman (1972) (with obvious changes for a vortex having a discontinuous vorticity field, with zero vorticity for $r > a_0$). This derivation depends on the utilization of a Biot-Savart formula with cutoff, and is determined by matching this to a special vortex filament configuration (the helical vortex). The present derivation seems to us to be on more fundamental ground.

We note that waves of extreme length ($k\to 0$) have zero phase and group velocity relative to the fluid at infinite distances from the axis of rotation. Furthermore, as pointed out by Maxworthy *et al.* (1985), the phase and group velocities are in the same direction and the ratio of phase speed, c, to group speed, C_g, is $\frac{1}{2}$ in the limit $k\to 0$.

Figure 1 shows a comparison between the asymptotic formula (41) and numerical computations of the dispersion relation for the slow branch. The results shown are for the vortex (18) with $W_0 = 0$, $\alpha_2 = 1$, $\Gamma = 2\pi$, and $m = -1$ (the corresponding result for $m = 1$ is found by putting $\omega\to-\omega$). The agreement is acceptable for $|k|$ up to about 0.2 and very good for $|k| < 0.15$. Figure 2 gives the frequency, phase speeds and group speed for the same case, according to direct numerical computation. For flows with $W' \neq 0$, the dispersion relation is no longer exactly symmetric in k. Figures 3 and 4 compare (41) with results from a numerical integration of the HGE for the flow (18) with $W_0 = 0.4$, $\alpha_1 = 0.54$, $\alpha_2 = 1.28$, $\Gamma = (1.39)\,2\pi$, which Maxwothy *et al.* (1985) cite as a fit to their experimental data. Figure 5 is like figure 2, but for the case shown in figure 3.

Figure 6 shows the eigenfunctions determined by numerical integration and by the composite asymptotic result (42) for the case shown in figure 1, with $k = 0.10$. Two curves are shown but lie one on top of the other. For smaller k, the results are even more accurate.

4. Long fast waves and centre-modes: results for general vortices

The analysis of the previous section yields only one branch of the dispersion relation. It appears incapable of capturing the fast-wave modes. Our numerical experiments have shown that the eigenfunctions for fast modes, rather than decaying away from the axis on a radial scale comparable to the vortex core, decay much faster. Thus, in addition to the scales a_0/k and a_0, there is yet another and smaller scale of importance here. In fact, the eigenfunctions are concentrated in the region of the vortex axis that shrinks to zero as $k\to 0$. This suggests that these modes are centre-modes, similar to the weakly unstable centre modes recently studied by Stewartson & Brown (1985). An early hint that this might be so is given by Leibovich & Ma's (1983) computations showing that $\omega\to m\Omega(0)$ as $k\to 0$ so $\gamma(0)\to 0$: this is the identifying characteristic of the centre-modes of Stewartson & Brown. Their description requires a radial stretching focusing on the vicinity of the axis, the rapid variations being forced by the small minimum in $|\gamma(0)|$ achieved by centre-modes for $|\beta| \ll 1$.

Away from the axis, (22) still holds, when $|m| = 1$, but now there is no compelling reason to set the second term, which is singular as $r \to 0$, to zero. This forms the outer solution for the $|m| = 1$ centre-modes, but it is not necessary to restrict ourselves now to the case $|m| = 1$. It turns out, in fact, that the first few terms in the inner expansion do not require the detailed behaviour of the outer solution, but only its general form, and the latter can be specified *a priori* by a Frobenius series for arbitrary m.

It is convenient to consider

$$y = r\chi \tag{44}$$

as dependent variable in the HGE (20), in which case it assumes the form

$$r\frac{\mathrm{d}}{\mathrm{d}r}\frac{ry'}{1+\beta^2r^2}-\left[m^2+\frac{\bar{a}}{\bar{\gamma}}+\frac{\bar{b}}{\bar{\gamma}^2}\right]y = 0, \tag{45}$$

where $\bar{a}, \bar{b}, \bar{\gamma}$ have already been defined, and (for all m)

$$y(0) = y(\infty) = 0, \tag{46}$$

and we allow m to be any integer. To force out the leading approximation to the eigenvalue ω and eigenfunction y, we need only consider the neighbourhood of $r = 0$. The appropriate stretching will emerge in the course of the analysis. Centre-modes are characterized by $\bar{\gamma}(0) \to 0$, so we chose ω to make this condition hold to lowest order, and write

$$\sigma \equiv \omega/m = \Omega_0+\beta W_0-\tfrac{1}{2}(\Omega_2+\beta W_2)\,\Lambda(\beta), \tag{47}$$

where we have written

$$\Omega_0 = \Omega(0), \quad \Omega_2 = \Omega''(0), \quad W_0 = W(0), \quad W_2 = W''(0).$$

We now regard the number $\Lambda(\beta)$, assumed to vanish as $|\beta| \to 0$, as replacing ω (or σ) as the eigenvalue, and it will simultaneously determine the stretching of r. We will find that $\Lambda \propto |\beta|$ as $|\beta| \to 0$ unless $W_2 = 0$, in which case $\Lambda \propto \beta^2$. We also note here that regular neutral centre-modes are possible in the vicinity of wavenumbers with $\beta = W_2/2\Omega_0$, which is not generally small, and we shall return at the end of this section to consider these modes.

The functions $\bar{a}$, $\bar{b}$ and $\bar{\gamma}$ may be expanded in Taylor series near $r = 0$ with leading terms (temporarily retaining terms which vanish as $\beta \to 0$)

$$\bar{a}(r) = 2r^2[\beta W_2+2\Omega_2-2\beta^2\Omega_0], \tag{48a}$$

$$\bar{b}(r) = -2\beta r^2\Omega_0(2\beta\Omega_0-W_2), \tag{48b}$$

$$\bar{\gamma}(r) = \tfrac{1}{2}(\beta W_2+\Omega_2)\,(r^2+\Lambda). \tag{48c}$$

We now stretch r by taking

$$r^2 = \Lambda s, \quad y(r) = Y(s), \tag{49}$$

assuming $0 < \Lambda \ll 1$ (the case with $\Lambda < 0$ leads to unstable centre-modes with critical layers, and is assumed, based on the work of Stewartson & Brown (1985), not to be possible near $\beta = 0$), and substitute (49) into (45). To leading order, (45) becomes

$$s^2Y''+sY'+\left[-\frac{m^2}{4}+\frac{2\beta^2\Omega_0-2\Omega_2-\beta W_2}{\Omega_2+\beta W_2}\frac{s}{1+s}+\frac{2\beta\Omega_0(2\beta\Omega_0-W_2)}{\Lambda(\Omega_2+\beta W_2)^2}\frac{s}{(1+s)^2}\right]Y = 0. \tag{50}$$

This equation determines the structure of the eigenmodes near $r = 0$, and the possible values of Λ describe, according to (47), the leading approximation to the frequency ω.

We assume that $\Omega_2 \neq 0$ and let $\beta \to 0$. There are then two cases to consider: $W_2 \neq 0$, and $W_2 = 0$. In the first case, we take $\Lambda \propto |\beta|$ and in the second we take $\Lambda \propto \beta^2$. Both can be treated simultaneously if we retain both factors in the expression $(2\beta\Omega_0 - W_2)$ in the last term of (50). We therefore introduce an order β^0 parameter μ through the relation

$$\mu(\mu+1) = \frac{2\beta\Omega_0(2\beta\Omega_0 - W_2)}{\Omega_2^2 \Lambda}, \tag{51}$$

and (50) becomes (as $\beta \to 0$)

$$s^2 Y'' + s Y' + \left[-\frac{m^2}{4} - 2\frac{s}{s+1} + \mu(\mu+1)\frac{s}{(s+1)^2} \right] Y = 0. \tag{52}$$

This equation is reducible to a hypergeometric equation, with solutions vanishing at the origin in the form

$$Y = \frac{\hat{K} s^{\frac{1}{2}|m|}}{(1+s)^{\mu}} F(\hat{a}; \hat{b}; \hat{c}; -s) \tag{53a}$$

for $\mu > 0$, where F is the hypergeometric function and $\hat{a}$, $\hat{b}$, and $\hat{c}$ are given in terms of μ and $|m|$ by

$$\hat{a} = -\mu + \tfrac{1}{2}|m| - \tfrac{1}{2}(m^2+8)^{\frac{1}{2}}, \quad \hat{b} = -\mu + \tfrac{1}{2}|m| + \tfrac{1}{2}(m^2+8)^{\frac{1}{2}}, \quad \hat{c} = 1 + |m|. \tag{53b}$$

The procedure of Stewartson & Brown (1985) applies here without change for $|m| \neq 1$. We match (53a) to the solution of the Howard–Gupta equation with σ replaced by its leading-order term, $\sigma_0 = \Omega_0$. This has a Frobenius series about the origin in the form $A_0 r^p + A_1 r^{-p}$ where

$$p = (m^2+8)^{\frac{1}{2}}$$

provided p is not an integer. In the latter case, which arises only for $m^2 = 1$ and yields $p = 3$, the solution is of the form

$$A_0 r^3 + A_1[r^{-3} + b_0 r^3 \ln r]. \tag{54}$$

The match between the outer and inner solution is possible only if $\hat{b}$ is chosen to be a negative integer or zero. This determines μ as

$$\mu = \mu(M, m) = \tfrac{1}{2}(|m| + (m^2+8)^{\frac{1}{2}}) + M \quad (M = 0, 1, 2, \ldots). \tag{55}$$

Furthermore, the hypergeometric series then terminates and becomes a polynomial of degree M. We may now solve for Λ from (51). Since the transformation from r to s has a real inverse only if $\Lambda > 0$, an acceptable solution is possible only if the condition $\beta\Omega_0(2\beta\Omega_0 - W_2) > 0$ is met. The frequency ω now follows from (47), (51) and (55), and is given to this order by (assuming $\Omega_2 \neq 0$ and replacing β by k/m)

$$\omega = m\Omega_0 + kW_0 + \frac{k\Omega_0(mW_2 - 2k\Omega_0)}{m\Omega_2(\frac{1}{2}|m| + \frac{1}{2}(m^2+8)^{\frac{1}{2}} + M)(\frac{1}{2}|m| + (M+1) + \frac{1}{2}(m^2+8)^{\frac{1}{2}})} \tag{56a}$$

with

$$k\Omega_0(2k\Omega_0 - mW_2) > 0, \tag{56b}$$

where $\mu(M,m)$ has been written out in full. If $W_2 = 0$, condition (56*b*) is automatic. If $W_2 \neq 0$ and $k \to 0$, we may replace (56*b*) by

$$-km\Omega_0 W_2 > 0. \tag{56c}$$

The term of $O(k^2)$ is accurate only if $W_2 = 0$, and should be included in (56*a*) only in that case. Note, first, that this formula holds for all integral values of $|m|$, and is not restricted to bending waves. Second, the phase velocity of long waves is

$$c(k) = \frac{\omega}{k} = \frac{m\Omega_0}{k} + W_0 + \frac{W_2\Omega_0}{\Omega_2\overline{M}(M,m)} - \frac{2\Omega_0^2}{m\Omega_2\overline{M}(M,m)}k, \tag{57}$$

where the mode parameter $\overline{M}(M,m)$ is defined to be

$$\begin{aligned}\overline{M}(M,m) &= \mu(M,m)\,\mu(M+1,m)\\ &= (\tfrac{1}{2}|m| + \tfrac{1}{2}(m^2+8)^{\frac{1}{2}} + M)\,(\tfrac{1}{2}|m| + \tfrac{1}{2}(m^2+8)^{\frac{1}{2}} + M + 1)\end{aligned}$$

and the group velocity

$$C_g(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k} = W_0 + \frac{W_2\Omega_0}{\Omega_2\overline{M}(M,m)} - \frac{4\Omega_0^2 k}{m\Omega_2\overline{M}(M,m)}, \tag{58}$$

and these results are applicable only for waves satisfying (56*b*). As anticipated, $c \to \infty$ as $|k| \to 0$ for these fast waves. This is in contrast to the slow bending waves, for which $c(0) = 0$; in further contrast, the group velocity for $k = 0$, while finite, is non-zero for fast waves unless W_0 and W_2 both vanish (which of course they do for the important special case $W(r) \equiv 0$). Furthermore, if, as will be the case for flows with jet-like axial velocity distribution with $\Omega_2 < 0$, $W_2 < 0$ when $W_0 > 0$, the higher modes of fast waves have smaller group speed than the lower modes.

Eigenfunctions corresponding to various modes, M, are identified, as in Sturm–Liouville theory, by the number of their zeros. Thus the primary mode, having $M = 0$, has no zeros, $M = 1$ has one zero, $M = l$ has l zeros, and so forth: the hypergeometric functions involved are polynomials of degree M. The primary eigenfunction near $r = 0$ has the form

$$Y^{(0)}(s) = \frac{s^{\frac{1}{2}|m|}}{(1+s)^{\mu(0,m)}}, \tag{59}$$

which, in the case of $|m| = 1$ reduces to

$$Y^{(0)}(s) = \frac{s^{\frac{1}{2}}}{(1+s)^2}. \tag{60a}$$

Note that the scaling of the radial variable depends on both mode number M and $|m|$, since $r^2 = As$, and A depends on these parameters. Thus, for $M = 0$, $|m| = 1$, the case above, and assuming for definiteness that $W_2 \neq 0$,

$$r^2 = As, \quad A = -\frac{1}{3}\frac{\Omega_0 W_2}{\Omega_2^2}\beta. \tag{60b}$$

For the second eigenfunction, we have a slightly different scaling factor; thus for $|m| = 1$, $M = 1$,

$$Y^{(1)}(s) = \frac{s^{\frac{1}{2}}(1-2s)}{(1+s)^3}, \quad r^2 = As, \quad A = -\frac{1}{6}\frac{\Omega_0 W_2}{\Omega_2^2}\beta. \tag{61}$$

Figure 7 shows the first five branches of the dispersion relation for the fast branch

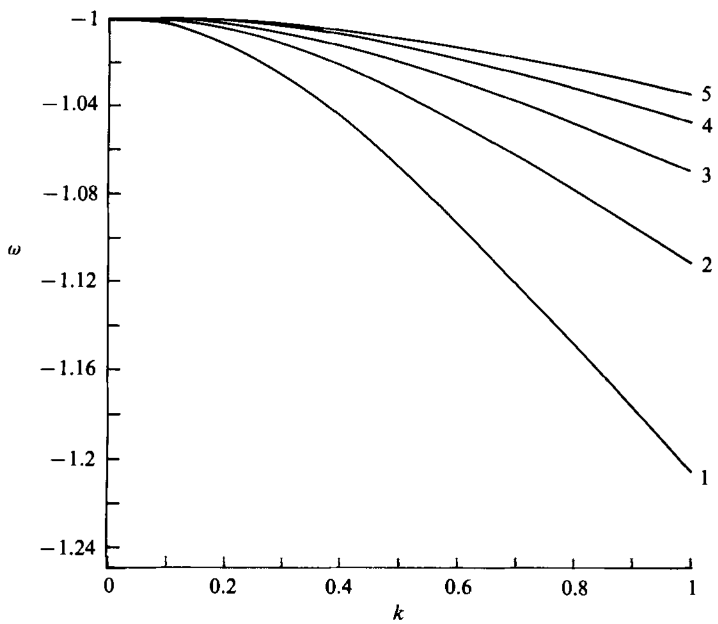

FIGURE 7. First five branches of the dispersion relation for fast modes for the flow A and $m = -1$. Results are from direct numerical computation. The portions of the curves for $k < 0.1$ are inferred (not computed).

and $m = -1$, for the flow (18) with $\alpha_2 = 1$, $W_0 = 0$, $\Gamma/2\pi = 1$, obtained by direct numerical solution of the Howard–Gupta eigenproblem. Figure 8 shows the corresponding eigenfunctions for $k = 0.01$. Figure 9 gives the first two modes for the flow (18) with $W_0 = 0.4$, $\Gamma/2\pi = 1.39$, $\alpha_1 = 0.54$, $\alpha_2 = 1.28$, showing comparison between results from direct numerical computation and the asymptotic eigenfunctions, for $k = 0.1$ and $k = 0.2$. A comparison between the numerically determined eigenvalues for the first five modes and the asymptotic formula (56*a*) is given in table 1 for flow (18) with $W_0 = 0$, $\alpha_2 = 1$, $\Gamma/2\pi = 1$ for $k = 0.2$, $m = -1$, corresponding to figure 7. Another kind of comparison is given in table 2 for the flow of figure 9. In all cases, the agreement between asymptotic formulae and numerical computations is good.

4.1. *Higher approximations*

We may proceed further with the expansion of the eigenvalues and eigenfunctions. Rather than do this for the general case, we illustrate by considering the examples (18) with $W_0 = 1$, $\alpha_1 = \alpha_2 = 1$, and $|m| = 1$. For this example the condition (56*b*), which must be satisfied if centre-modes are to exist, is $k\Gamma/m > 0$. For the primary mode with eigenfunction (59), the eigenvalue (56*a*) is

$$\omega = \frac{m\Gamma}{2\pi} + \frac{4k}{3}, \quad |m| = 1, \quad \frac{k\Gamma}{m} > 0, \tag{62}$$

where we note that the $O(k^2)$ term (proportional to Ω_0) is not applicable because $W_2 \neq 0$. To find the $O(k^2)$ term, we must find A to $O(\beta^2)$. This is done by keeping the terms of relative order β that have so far been neglected in (50), and insisting that the correction to the eigenfunction vanish as $s \to \infty$, as required by the matching.

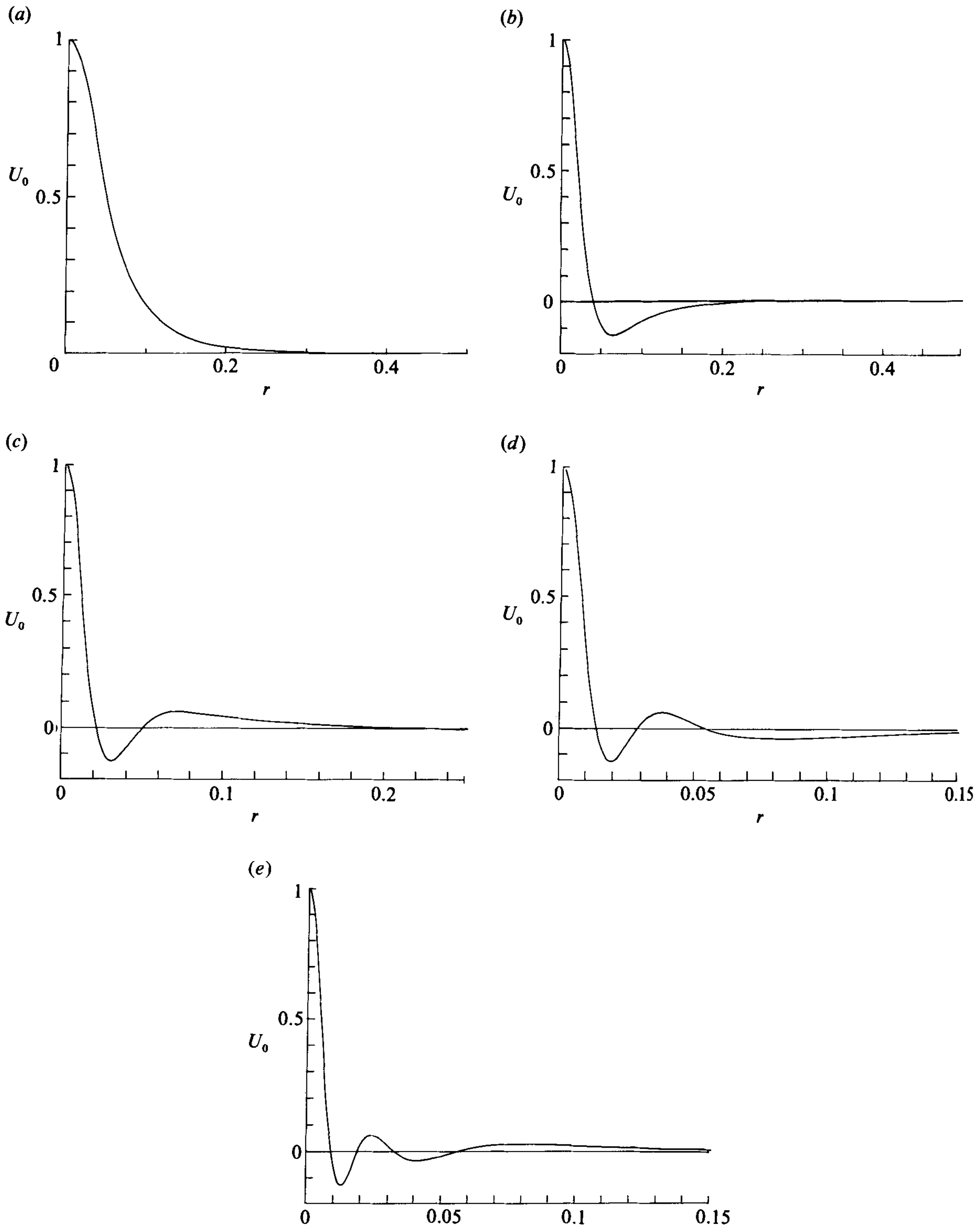

FIGURE 8. Eigenfunctions for the first five modes for $k = 0.01$ for the example of figure 7. (a) Primary mode, (b) mode 2, (c) mode 3, (d) mode 4, (e) mode 5.

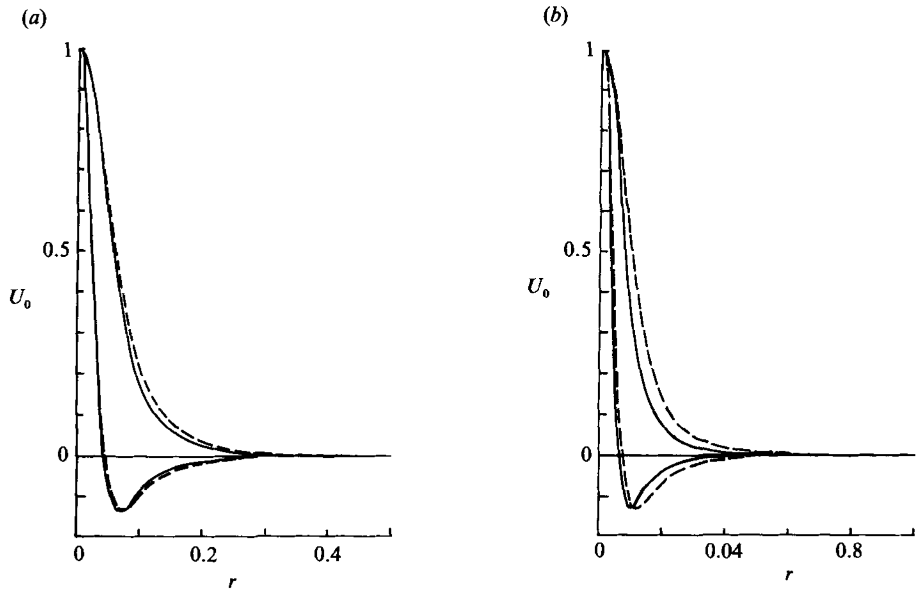

FIGURE 9. (*a*) First two fast $m = 1$ eigenfunctions for the flow B with $k = 0.1$, showing comparison between direct numerical computation and the asymptotic eigenfunctions. (*b*) As in (*a*) but for $k = 0.2$.

M	ω, numerical	ω, analytical
0	−1.012938	−1.013333
1	−1.006514	−1.006666
2	−1.003931	−1.004000
3	−1.002637	−1.002666
4	−1.001899	−1.001905

TABLE 1. Frequency ω at $k = 0.2$ for the flows of figure 7 for the first five modes, comparing the results of direct numerical computation to the asymptotic results of (56*a*)

The relations (48) must be extended by one further term, and we now write

$$\bar{a}(r) = 4As\left[\frac{\Gamma}{2\pi}(1-As)+\beta\right],$$

$$\bar{b}(r) = -4\beta\frac{\Gamma}{2\pi}As\left(1+\frac{\beta\Gamma}{2\pi}-\frac{3As}{2}\right),$$

$$\bar{\gamma}(r) = -mA\left[\beta s+\frac{\Gamma}{4\pi}\left(1+s-\frac{As^2}{3}\right)\right].$$

To the required order, (45) now becomes

$$s^2Y''+sY'+\left\{-\frac{m^2}{4}-\frac{2s}{1+s}\left[1-As+\frac{2\pi\beta}{\Gamma}-\frac{4\pi\beta s}{\Gamma(1+s)}+\frac{As^2}{3(1+s)}\right]\right.$$
$$\left.+\frac{8\pi\beta}{\Gamma A}\frac{s}{(1+s)^2}\left[1+\frac{\beta\Gamma}{2\pi}-\tfrac{3}{2}As-\frac{8\pi\beta s}{\Gamma(1+s)}+\frac{2As^2}{3(1+s)}\right]\right\}Y = 0. \quad (63)$$

k	ω, numerical	ω, asymptotic
0.15	1.195	1.196
0.25	1.319	1.323
0.35	1.440	1.448

TABLE 2. Eigenvalues for the $m = 1$ primary fast branch for flow (18) with $W_0 = \Gamma/2\pi = \alpha_1 = \alpha_2 = 1$, comparing results of direct numerical calculation with the asymptotic formula (66)

In (63) we now set

$$\left.\begin{aligned} Y(s) &= Y_0(s) - \beta Y_1(s), \\ A &= \tfrac{2}{3}(2\pi\beta/\Gamma) + A_1(2\pi\beta/\Gamma)^2, \end{aligned}\right\} \tag{64}$$

where $Y_0(s)$ has previously been found. The equation for $Y_1(s)$ is

$$\begin{aligned} s^2 Y_1'' + s Y_1' &+ \left(-\frac{1}{4} - \frac{2s}{1+s} + \frac{6s}{(1+s)^2}\right) Y_1 \\ &= 2 Y_0 \frac{s}{1+s}\left\{\tfrac{2}{3}s - 1 + \frac{s-1}{s+1} - \frac{2}{9}\frac{s^2}{1+s}\right. \\ &\qquad \left. - \frac{3}{1+s}\left(-\left(\frac{\Gamma}{2\pi}\right)^2 + s + \frac{4s}{1+s} - \frac{4}{9}\frac{s^2}{1+s}\right) - \frac{9}{2}\frac{A_1}{1+s}\right\}. \end{aligned} \tag{65}$$

If the right-hand side of this equation is denoted $Y_0(s)\,X(s)$, then it is easy to show that $Y_1(s)$ is finite at the origin and decays at infinity only if

$$\int_0^\infty \frac{X(s)\,Y_0^2(s)}{s}\,\mathrm{d}s = 0.$$

This condition finally reduces to

$$A_1 = \frac{2}{3}\left(\frac{\Gamma}{2\pi}\right)^2 - \frac{10}{9},$$

so that (62) is now corrected to read

$$\omega = m\frac{\Gamma}{2\pi} + \tfrac{4}{3}k + \frac{m\Gamma}{6\pi}k^2\left[1 - \frac{5}{3}\left(\frac{2\pi}{\Gamma}\right)^2\right]. \tag{66}$$

If instead of setting $W_0 = 1$ in (18) we keep it as an arbitrarily prescribed parameter, (66) is replaced by

$$\omega = m\frac{\Gamma}{2\pi} + \tfrac{4}{3}kW_0 + \frac{m\Gamma}{6\pi}k^2\left[1 - \frac{5W_0^2}{3}\left(\frac{2\pi}{\Gamma}\right)^2\right], \tag{67}$$

with the requirement that $|m| = 1$ and $W_0 k\Gamma/m > 0$. For the second mode, a similar argument leads to

$$\omega = \frac{m\Gamma}{2\pi} + \tfrac{7}{6}kW_0 + \frac{1}{12\pi}m\Gamma k^2\left(1 - \frac{11}{6}\left(\frac{2\pi W_0}{\Gamma}\right)^2\right). \tag{68}$$

When $W_0 = 0$, (67) and (68) agree with the results (60*b*) and (61) for A with $\mu = 2$ and 3 respectively. In these cases, it is not difficult to extend the dispersion relations for $\mu = 2$ to include a further term in β. The result applies to the primary mode and is

$$\omega = m\Omega_0 - \tfrac{1}{3}m\frac{\Omega_0^2}{\Omega_2}k^2 - \tfrac{1}{27}m\frac{\Omega_0^2}{\Omega_2}k^4\left(\frac{5\Omega_2}{\Omega_0} + \frac{6\Omega_0}{\Omega_2} - \frac{\Omega_0^2\Omega_4}{\Omega_2^3}\right), \tag{69}$$

for $|m| = 1$, arbitrary $\Omega(r)$, but $W_0 = 0$. The first logarithmic term to appear in (69) is $k^8 \log(1/|k|)$.

4.2. *Short-wave centre-modes*

The search for centre-modes has hinged upon the smallness of the quantity

$$\beta\Omega_0[2\beta\Omega_0 - W_2],$$

occurring in (48*b*). This is small for β small, the cases explored so far, but it can also be made small for $2\beta\Omega_0 - W_2$ small, which for $W_2 \neq 0$ will be at values of β that are not small. Centre-modes therefore might exist for values of β near $W_2/2\Omega_0$, which corresponds to the region in which (for certain velocity profiles) Stewartson & Brown (1985) found weakly unstable centre-modes.

To explore the possibility of neutral cenre-modes in this wavenumber range, we set $\beta = W_2/2\Omega_0$ in (52), except for the factor $(W_2 - 2\beta\Omega_0)/\Lambda$. To leading order the equation becomes

$$s^2 Y'' + sY' + \left\{-\frac{m^2}{4} - \frac{\tilde{\lambda}s}{s+1} + \tilde{\mu}\frac{(\tilde{\mu}+1)s}{(1+s)^2}\right\} Y = 0,$$

where

$$\tilde{\lambda} = \frac{4\Omega_0\Omega_2}{2\Omega_0\Omega_2 + W_2^2},$$

$$\tilde{\mu}(\tilde{\mu}+1) = \frac{W_2(2\beta\Omega_0 - W_2)}{(\Omega_2 + W_2^2/2\Omega_0)^2 \Lambda}.$$

The analysis then follows the plan previously employed and succeeds provided

$$-1 \leqslant W_2^2 - 2\beta\Omega_0 W_2 < 0. \tag{70}$$

If this condition is satisfied, the dispersion relation is

$$\omega = m\Omega_0 + \frac{mW_0 W_2}{2\Omega_0} + \frac{\frac{1}{2}mW_2^2 - k\Omega_0 W_2}{(\Omega_2 + W_2^2/2\Omega_0)\tilde{\mu}(\tilde{\mu}+1)}, \tag{71}$$

where

$$\tilde{\mu} = \tfrac{1}{2}(|m| + (m^2 + 4\tilde{\lambda})^{\frac{1}{2}}) + M \quad (M = 0, 1, 2, \ldots). \tag{72}$$

5. Solitons and soliton windows

5.1. *Solitons and interpretation of experimental data*

The nonlinear Schrödinger equation (15) has solutions representing solitons (and thus leading to persistent motion for long time intervals) provided the product $\hat{\mu}\hat{\nu} > 0$ (Whitham 1974), where $\hat{\mu}$ and $\hat{\nu}$ are the coefficients appearing in (15). If this condition does not hold, then an initial disturbance having an envelope that vanishes at infinity will ultimately decay due to dispersion. The single soliton solution of (15) takes the form of a long envelope modulation of shorter waves and may be written as

$$A(Z,\tau) = A_0\, \mathrm{e}^{\mathrm{i}\Phi} \operatorname{sech}\left\{\mathrm{A}_0\left(\frac{\hat{\nu}}{2\hat{\mu}}\right)^{\frac{1}{2}}(Z - C\tau) + \Delta\right\}, \tag{73}$$

where

$$\Phi = \frac{C}{2\hat{\mu}}Z + \left(\frac{A_0^2\hat{\nu}}{2} - \frac{C^2}{4\hat{\mu}}\right)\tau + \Phi_0. \tag{74}$$

The real parameters A_0 (positive), C, Δ, and Φ_0 are determined by initial conditions. Of these, Δ and Φ_0 represent the invariance of (15) to shifts in the origin of Z and τ, while C and A_0 depend upon the spatial form of the initial disturbance.

In terms of our original space and time variables, the velocity vector to $O(\epsilon)$ is

$$\boldsymbol{u}(r,\theta,t) = \boldsymbol{U}(r)+A_0\,\epsilon\hat{\boldsymbol{u}}_0(r)\exp \mathrm{i}(\bar{k}z+m\theta-\bar{\omega}t)\,\mathrm{sech}\,[\epsilon A_0(\hat{\nu}/2\hat{\mu})^{\frac{1}{2}}(z-\bar{C}t)], \tag{75}$$

where

$$\left.\begin{aligned} \bar{C} &= C_{\mathrm{g}}+\epsilon C, \\ \bar{k} &= k_0+\frac{\epsilon C}{2\hat{\mu}}, \\ \bar{\omega} &= \omega_0+\frac{\epsilon C C_{\mathrm{g}}}{2\hat{\mu}}+\epsilon^2\left[\frac{C^2}{4\hat{\mu}}-\frac{A_0^2\,\hat{\nu}}{2}\right]. \end{aligned}\right\} \tag{76}$$

where we have set $\Delta = \Phi_0 = 0$. Thus the nonlinear effects result in a correction to the wavenumber, changing it by an $O(\epsilon)$ amount from the wavenumber k_0 of the carrier wave to $\bar{k}$, a correction of the frequency from the carrier frequency ω_0 to $\bar{\omega}$, and a correction of the group velocity from the carrier value C_{g} to $\bar{C}$.

Notice that given the linear dispersion relation, $\omega(k)$, and the form of k,

$$\omega(\bar{k}) = \omega(k_0)+\frac{\epsilon C}{2\hat{\mu}}\frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)+\left(\frac{\epsilon C}{2\hat{\mu}}\right)^2\frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}(k_0), \tag{77}$$

so that in (76)

$$\bar{\omega} = \omega(\bar{k})-\tfrac{1}{2}\epsilon^2A_0^2\,\hat{\nu}, \tag{78}$$

and

$$\begin{aligned} \frac{\mathrm{d}\omega}{\mathrm{d}k}(\bar{k}) &= \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)+\frac{\epsilon C}{2\hat{\mu}}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}(k_0) \\ &= C_{\mathrm{g}}+\epsilon C = \bar{C}. \end{aligned} \tag{79}$$

Thus the velocity of the modulation envelope corrected for nonlinear effects is just the linear group velocity associated with wavelets of wavenumber $\bar{k}$. The only essential nonlinear content of the solution form (75) is then the envelope shape, with the sech functional behaviour and a length that is inversely proportional to the disturbance amplitude ϵA_0, and a nonlinear frequency correction proportional to $(\epsilon A_0)^2$.

These considerations show that a linear interpretation of experimental observations should be very good provided the amplitude parameter, ϵA_0, is small. In particular, measurement of the length of wavelets propagating through the group yields $\bar{k}$, and measurement of the speed of the group gives the linear group speed, or the slope of the linear dispersion relation $\omega(\bar{k})$. A measurement of the phase speed of wavelets $\bar{c}(\bar{k})$ gives

$$\bar{c}(\bar{k}) = \frac{\bar{\omega}}{\bar{k}} = \frac{\omega(\bar{k})}{\bar{k}}-\frac{(\epsilon A_0)^2\,\hat{\nu}}{2\bar{k}}.$$

Thus the dispersion relation, $\omega(\bar{k})$ can be determined from the phase speed and length of wavelets provided the measurement yields speeds that are not too small. For long waves, or $\bar{k}$ small, phase speed measurements are not reliable unless $(\epsilon A_0)^2/\bar{k}$ is small. If this is not the case, one could attempt to determine $\omega(\bar{k})$ by measurement of the group speed and constructing ω from it. It turns out that in our case the nonlinear Schrödinger equation itself fails as $k\to 0$, so that we cannot use results obtained from it as guidance to interpretation of experiments. The failure of the nonlinear Schrödinger equation as $k\to 0$ is discussed in §6.

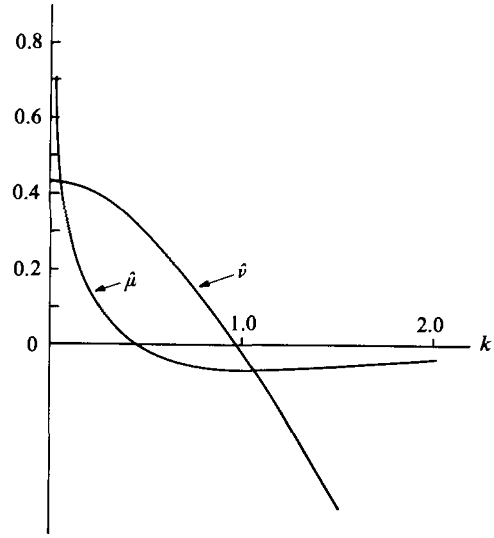

FIGURE 10. Nonlinear Schrödinger equation coefficients for case A discussed in the text.

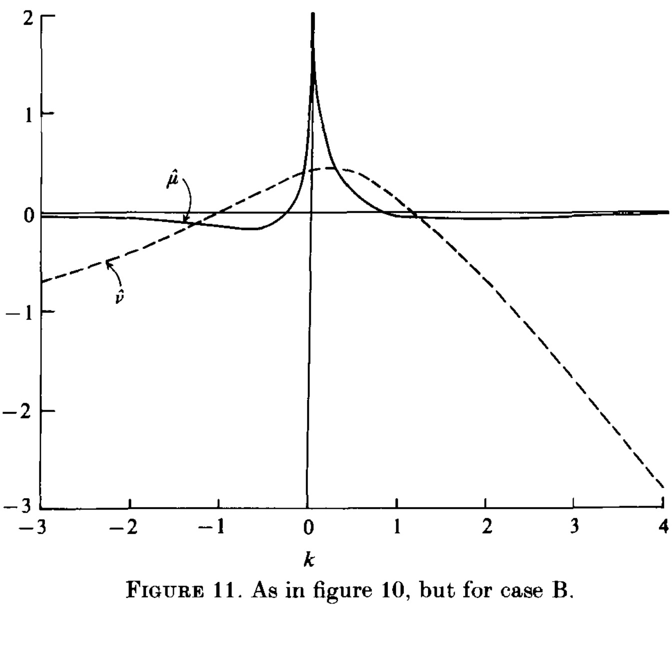

FIGURE 11. As in figure 10, but for case B.

5.2. *Soliton windows: numerical results on the slow branch*

The coefficients $\hat{\mu}$ and $\hat{\nu}$ in (15) are functionals of the columnar vortex velocity vector $\boldsymbol{U}(r)$ and depend parametrically on the carrier wavenumber k_0. For a given vortex $\boldsymbol{U}(r)$, carrier waves may or may not admit solitons for a prescribed value of k_0, since solitons are possible only if $\hat{\mu}\hat{\nu} > 0$. For example, Leibovich & Ma (1983) searched for solitons on the primary fast branch for $|m| = 1$ for the profile (18) with $\alpha_2 = 1$, $W_0 = 0$, $\Gamma = 2\pi$, and found that solitons were excluded for $|k| \leqslant k_1$ and $|k| \geqslant k_2$ with

$$k_1 \approx 0.68 \quad \text{and} \quad k_2 \approx 1.$$

Solitons were possible in the 'window' $k_1 < k < k_2$.

We have repeated this search for $|m| = 1$ but this time exploring the slow branch. Two examples have been examined, both special cases of (18). One example, case A, is the vortex with $W_0 = 0$, $\Gamma = 2\pi$, previously studied by Leibovich & Ma. The second, case B, has $\alpha_1 = 0.54$, $\alpha_2 = 1.28$, $W_0 = 0.4$, $\Gamma = (1.39)\,2\pi$, and was found by Maxworthy *et al.* (1985) to fit data in their experiments.

Numerical results for the nonlinear Schrödinger equation coefficients $\hat{\mu}$ and $\hat{\nu}$ for $m = -1$ as functions of wavenumber, are plotted in figures 10 and 11 for cases A and B respectively.

There is a soliton window for case A for $|k| < 0.45$ and a second window for $|k| > 0.96$. Results for cases with $W \equiv 0$, as in case A, are symmetric in k as noted earlier by Leibovich & Ma. For case B, soliton windows exist in $-0.3 < k < 0.8$, in $k > 1.2$ and in $k < -1$. For both cases longwave and shortwave solitons are possible: there are, however, bands of intermediate wavelengths in which solitons are excluded.

6. Failure of the nonlinear Schrödinger equation for very long slow waves

If we consider carrier waves with wavenumber $k_0 \rightarrow 0$, then the previous analysis breaks down in a crucial way, which is associated with the non-analyticity of the linear dispersion relation at $k = 0$. It is of course true that the nonlinear Schrödinger equation (probably) always needs to be formally replaced in this limit, but when the dispersion relation is analytic the issue is incidental as shown by Davey (1972). For a conservative system with $\omega(k)$ analytic at $k = 0$, $\hat{\mu} = 0$. As Davey shows the appropriate replacement is the Korteweg–de Vries (KdV) equation but the close connections of the solitary wave solutions of the KdV and NLS stressed by Davey shows that there are no essentially new features arising.

In the present case, the slow branch dispersion relation has a singularity at $k = 0$, and for $|m| = 1$ and small k_0 it gives

$$\hat{\mu}(k_0) = -\frac{m\Gamma}{4\pi}\left(\ln\left(\frac{2}{|k_0|}\right) - \tfrac{3}{2} + K - \gamma_e\right), \tag{80}$$

as may be seen from (41). Thus $\hat{\mu}(k_0) \rightarrow \infty$ as $k_0 \rightarrow 0$, although it does so slowly, and the consequences are more interesting.

The perturbation procedure leading to the NLS contemplates $\hat{\mu} = O(1)$ as $\epsilon \rightarrow 0$ uniformly in k_0. In point of fact, the analysis assumes the validity of the Taylor series $\omega(k_0 + \epsilon\kappa) = \omega_0 + \epsilon C_g(k_0)\,\kappa + \epsilon^2\hat{\mu}(k_0)\,\kappa^2 + \ldots$, for $\kappa = O(1)$, in which case the spatial and temporal scales chosen are appropriate. Evidently this expansion fails to provide an appropriate local picture if $k_0 = \epsilon\kappa = O(\epsilon)$ when the carrier wave is as long as the modulation envelope, so that a breakdown may be expected for $|k_0| = O(\epsilon)$. The non-analyticity of the dispersion relation is due to the unboundedness of the fluid and to the irrotational nature of the motion for radial distances large compared to the vortical core, and would not arise for fluid contained in a tube with radius comparable to the core size (strictly speaking, any finite tube radius would do but the accuracy of approximation found presumably is severely affected if the tube is much larger than the core).

A similar failure of an amplitude equation was encountered by Leibovich (1970), and we can proceed in a similar way. The changes required begin in (9), where we take the linear phase ψ simply to be $m\theta$; thus we expand the velocity vector in a Fourier series in θ, but do not *a priori* assume normal-mode behaviour in z or time. To lowest order, the motion sought is independent of z and t, and represents a bodily displacement and rotation of the entire vortex column. Time evolution and axial spatial dependence are weak, and we therefore assume that the perturbation velocity vector depends on a slow axial variable

$$Z = \delta z,$$

with δ to be determined so that nonlinear terms can ultimately be balanced, and a sequence of slow times

$$\tau_1 = g_1(\delta, \epsilon)\, t, \quad \tau_2 = g_2(\delta, \epsilon)\, t, \quad \ldots,$$

where we suppose for the moment that δ and ϵ are independent small parameters with δ^{-1} measuring the initial axial length of the perturbations and ϵ their amplitudes, and the g_j are gauge functions vanishing as either δ or ϵ vanish. Now the velocity is

$$\boldsymbol{U}(r) + \epsilon \boldsymbol{u}(r, \theta, Z, \tau_1, \tau_2, \ldots, \delta, \epsilon),$$

and $\boldsymbol{u}$ is developed in a series of gauge functions vanishing with δ and ϵ. To lowest order

$$\boldsymbol{u} = \boldsymbol{u}_1 = \hat{\boldsymbol{u}}_1(r)\, \mathrm{e}^{\mathrm{i}m\theta} A(Z, \tau_1, \ldots),$$

with $|m| = 1$, where the radial component of $\hat{\boldsymbol{u}}_1(r)$ is $\hat{u}_1 = \Omega(r)$, which may be seen from the analysis of §3. Thus, the complete vector at lowest order is

$$\boldsymbol{u}_1 = A(\Omega(r), \mathrm{i}(r\Omega' + \Omega), \mathrm{i}W')\, \mathrm{e}^{\mathrm{i}\theta} + \text{c.c.},$$

where we have taken $m = 1$ with no further loss of generality. As in §3, we note that for large radial distances, the flow is irrotational to all algebraic orders in the small parameters and now the outer variables are Z and $\rho = \delta r$. In these variables, the velocity is described by a potential $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, and

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial Z^2} = 0.$$

Following Leibovich (1970), we work with the Fourier transform with respect to Z, written as

$$\bar{\phi} = \int_{-\infty}^{\infty} \phi(\rho, \theta, Z, T)\, \mathrm{e}^{-\mathrm{i}\alpha Z}\, \mathrm{d}Z.$$

Assuming
$$\phi = \sum_{m=-\infty}^{\infty} \phi_m(\rho, Z, t)\, \mathrm{e}^{\mathrm{i}m\theta},$$

we have
$$\phi_m(\rho, Z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_m(\alpha, t)\, K_{|m|}(|\alpha|\rho)\, \mathrm{e}^{\mathrm{i}\alpha Z}\, \mathrm{d}\alpha,$$

were K_m is the modified Bessel function of order m.

The analysis now follows the procedure used by Leibovich (1970). We omit the details, but note that consistent matches are required between the inner and outer approximations as in §3. In fact, the analysis of §3 produces the linear parts of the expansions required, and the dispersion relation found there gives, upon replacement of ω by $\mathrm{i}g_1(\delta, \epsilon)(\partial/\partial\tau_1)$, the Fourier transform of the linear contribution to the evolution equation for A. If the Fourier transform of A is $\bar{A}(\alpha, \tau_1, \tau_2, \ldots)$, then

$$\mathrm{i}g_1 \frac{\partial \bar{A}}{\partial \tau_1} + \frac{\Gamma}{4\pi}\delta^2\alpha^2\left[\ln\left(\frac{2}{\delta|\alpha|}\right) + K - \gamma_{\mathrm{e}}\right]\bar{A} = 0,$$

where K is defined in (37). The nonlinear terms resonant with the $\mathrm{e}^{\mathrm{i}\theta}$ component are unchanged in form from §2, and therefore lead to the need for a slow time τ_2 with

$$\mathrm{i}g_2 \frac{\partial \overline{A}}{\partial \tau_2} = -\epsilon^2 \hat{\nu}(0)\, \overline{A|A|^2},$$

where $\hat{\nu}(0)$ is the value of $\hat{\nu}$ evaluated at $k = 0$. Thus, in terms of the original time variable

$$\mathrm{i}\frac{\partial \overline{A}}{\partial t} + \frac{\Gamma}{4\pi}\delta^2\alpha^2\left[\ln\left(\frac{2}{\delta|\alpha|}\right) + K - \gamma_{\mathrm{e}}\right]\overline{A} + \epsilon^2\hat{\nu}(0)\,\overline{A|A|^2} = 0.$$

Let

$$\frac{\Gamma}{4\pi}\frac{\delta^2 \ln \delta^{-1}}{\epsilon^2} = \tilde{\tilde{\mu}},$$

and $\tau = \epsilon^2 t$, then

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \frac{\tilde{\tilde{\mu}}}{2\pi \ln \delta^{-1}}\int_{-\infty}^{\infty} \alpha^2\left[\ln\left(\frac{2}{\delta|\alpha|}\right) + K - \gamma_{\mathrm{e}}\right]\overline{A}\, \mathrm{e}^{\mathrm{i}\alpha Z}\, \mathrm{d}\alpha + \hat{\nu}(0)\, A|A|^2 = 0. \tag{81}$$

If we were to ignore terms $O(\ln|\alpha|/\ln\delta^{-1})$ as $\delta \to 0$, then the coefficient multiplying $\overline{A}$ in this equation would simplify to α^2, and we would revert to the NLS. It is well known, however, that in asymptotic approximations terms involving the logarithm of small quantities should not be separated from $O(1)$ terms. Furthermore, this would neglect the term $\alpha^2 \ln|\alpha|^{-1}$ which dominates the $\ln\delta^{-1}$ term for that portion of the integration interval for which $|\alpha| < \delta$. For both of these reasons, we retain the grouping as shown. Furthermore, as in Leibovich (1970) we note that

$$\ln\left(\frac{2}{\delta|\alpha|}\right) - \gamma_{\mathrm{e}} = K_0(|\alpha|\,\delta), \tag{82}$$

with an error of $O(L\ln L^{-1})$ as $L \to 0$ where $L = \delta|\alpha|$. There is, therefore, no loss in accuracy in making this replacement for the relevant terms in (81) (there is even the prospect that continuing the approximation procedure to higher orders would produce additional contributions agreeing with the series for K_0 for small argument). With this done, the equation for $A(Z,\tau)$ is

$$\mathrm{i}\frac{\partial A}{\partial \tau} - \frac{K\tilde{\tilde{\mu}}}{\ln\delta^{-1}}\frac{\partial^2 A}{\partial Z^2} - \frac{\tilde{\tilde{\mu}}}{2\ln\delta^{-1}}\frac{\partial^2}{\partial Z^2}\int_{-\infty}^{\infty}\frac{A(\xi,\tau)\,\mathrm{d}\xi}{((Z-\xi)^2+\delta^2)^{\frac{1}{2}}} + \hat{\nu}(0)\,|A|^2 A = 0. \tag{83}$$

Since the integral term is $O(\ln\delta^{-1})$ as $\delta \to 0$, the integral must be retained if $\tilde{\tilde{\mu}} = O(1)$, which shows that a balance with nonlinear terms occurs now for

$$\epsilon = \delta(\ln\delta^{-1})^{\frac{1}{2}}. \tag{84}$$

With this choice, the $\partial^2 A/\partial Z^2$ term in (83) is formally negligible in the limit $\delta \to 0$ but, for the reasons previously stated concerning the treatment of logarithmically small terms, we retain it.

Very long waves apparently are not determined entirely locally when the outer flow is irrotational and infinite. Instead, they are controlled by an integrodifferential equation that links local evolution with the motion of distant fluid elements in the core. That this general characteristic should emerge is perhaps not surprising, since the Biot–Savart formula for the motion of a vortex filament of zero core area has this feature.

Axisymmetric long waves in vortices embedded in an infinite irrotational fluid, a case considered by Leibovich (1970), lead to the same integral appearing in (83). The nonlinear term is $A\,\partial A/\partial Z$ rather than $|A|^3 A$. In his analysis of the connections between the NLS equation and the Korteweg–de Vries equation, Davey (1972) observes that the cubic nonlinearity and the product $A\,\partial A/\partial Z$ are connected, since the former term is an approximation to the latter for waves centred on a carrier wave of finite length. Here, although $k \rightarrow 0$, our waves are still centred on a wave of finite wavenumber, since the azimuthal wavenumber is not taken to be zero.

7. Conclusions

Leibovich & Ma found an $m = -1$ soliton window on the primary fast branch for case A ((18) with $W_0 = 0$, $\Gamma/2\pi = 1$, $\alpha_2 = 1$) in the approximate range $0.68 < k < 1$. We can compare with present results (figure 10) for the corresponding window for this flow for the slow branch. There are two of these; the longwave window occupies the approximate band $|k| < 0.45$ and clearly does not overlap with the fast branch window. The second, given approximately by $|k| > 0.96$ appears to overlap, but just barely. If the overlap is genuine, and not due to numerical errors, then it would indicate the possible propagation of two wave packets with different speeds, but centred about the same carrier wavenumber.

We have not established soliton windows on fast branches in this paper, and so we are unable to say whether there are overlaps between fast and slow branch windows for case B. This case was of course chosen because of its relationship with the experiments of MHR. We do not attempt here to analyse their experimental data in the light of present results. We note only that the range of wavenumbers for which 'kink' waves were observed seems to be roughly $0.25 \leqslant k \leqslant 0.35$. If the profiles of case B do properly fit their data, then this range falls within the slow branch soliton window, as may be seen from figure 11. Waves outside a soliton window are of course observable, they merely spread and ultimately decay due to dispersive effects. Waves within a window would be expected to be more robust, persist (as solitons) much longer, and therefore figure more prominently in observations.

Our analysis for long waves has revealed results for the wave propagation characteristics for arbitrary concentrated vortices, with or without axial velocities of arbitrary form. The explicit results obtained should be of general value for future investigations. The dispersion relations (41) and (56) allow us to infer simple rules about the geometry and motion of constant-phase surfaces of the slow and the fast waves. The intersection of a constant-phase surface with a cylinder at fixed radius r is the helix

$$kz + m\theta - \omega t = \text{constant}.$$

For the slow waves, we may substitute (41) to give

$$kz + m\theta + m\frac{\Gamma}{4\pi}k^2\left(\ln\left(\frac{2}{|k|}\right) + K - \gamma_e\right)t = \text{constant},$$

and $|m| = 1$. For all radial locations and fixed axial location, the constant-phase surface rotates slowly with rigid-body angular velocity

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\Gamma}{4\pi}k^2\left\{\ln\left(\frac{2}{|k|}\right) + K - \gamma_e\right).$$

For our discussions here, let us now adopt the convention that the orientation of the z-axis is chosen so that increasing z is in the direction of the axial component of vorticity of the basic flow. (The direction of the basic flow axial vorticity cannot change with r without violating Rayleigh's criterion for stability, and we therefore

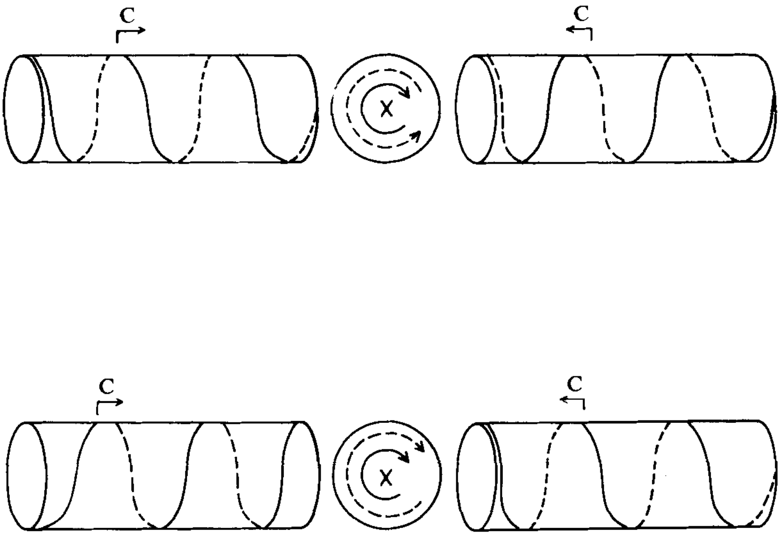

FIGURE 12. Top part of figure shows the geometry and motion of phase fronts for slow waves, and the bottom part for fast waves. In each case the drawing in the centre is a view along the vortex axis in the direction of the basic flow axial vorticity. The sense of rotation of fluid particles in the basic flow is shown in the centre figure by the solid curve, and the sense of rotation of phase fronts by the dashed curve. Slow waves propagating in the direction of the basic flow axial vorticity have phase fronts with a right-hand helix sense, those propagating in the opposite direction have a left-hand twist. For fast waves (lower panel), the sense of twist and propagation direction depend on the sign of $W''(0)$ (assumed in the figure non-zero); the left-hand drawing shows the case with $W''(0) < 0$, for which the twist is left-handed and the phase fronts advance in the direction of the basic flow axial vorticity. The right-hand figure shows the case $W''(0) > 0$, where the propagation direction and twist are reversed.

assume that the axial vorticity is one-signed.) Thus Ω_0 and Γ are by definition positive, and so slow waves rotate (slowly) in a sense *opposite* to that of the rotation of fluid particles in the basic vortex. Furthermore, since

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = -\frac{k}{m},$$

and
$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_{\text{phase}} = c(k) = -mk\frac{\Gamma}{4\pi}\left\{\ln\left(\frac{2}{|k|}\right) + K - \gamma_{\mathrm{e}}\right\},$$

waves which propagate in the direction of z-increasing have $mk < 0$ and hence $\mathrm{d}\theta/\mathrm{d}z > 0$; that is, phase fronts propagating in the direction of the basic flow axial vorticity have an intersection with a cylinder in the shape of a helix with a right-hand twist. Those propagating in the direction opposed to the basic axial vorticity have a left-hand twist ($\mathrm{d}\theta/\mathrm{d}z < 0$).

For long fast waves, the helix is determined by the conditions (56*a*) with $k \rightarrow 0$, or

$$kz + m\theta - m\Omega_0 t = \text{constant}.$$

All waves in the fast branches have phase fronts that rotate, at each radius r, with angular speed

$$\mathrm{d}\theta/\mathrm{d}t = \Omega_0,$$

that is, at a fixed axial station, a constant-phase surface rotates with the same angular speed as do fluid particles in the basic flow at that radial location. Phase surfaces move at the (fast) speed $(\mathrm{d}z/\mathrm{d}t)_{\text{phase}} = (m/k)\,\Omega_0$. If $W_2 = 0$, waves propagating in the direction of z-increasing have a left-hand helix sense, while those propagating in the direction of decreasing z have a right-hand helix sense. If $W_2 \neq 0$, the form is further restricted since (56*c*) must apply. There are two cases to consider: (i) $W_2 < 0$ and (ii) $W_2 > 0$. If $W_0 > 0$, (i) is jetlike while (ii) is wakelike near the axis, while if $W_0 < 0$, then (i) is wakelike while (ii) is jetlike. In case (i), condition (56*c*) admits only waves with $km > 0$ (or $k/m > 0$), so that phase surfaces propagate only in the direction of z-increasing and have a left-hand helix sense. In case (ii), only waves with $km < 0$ are allowed, and these propagate in the negative z-direction and have a right-hand helix sense. The helix sense in all fast-wave cases may be simply summed up: the helix sense is opposite to that of the vortex lines in the basic flow.

Figure 12 is a pictorial representation of the helix sense, and the propagation and rotation directions of slow and fast waves.

One line of enquiry of potential interest is the question of interactions between modes obtained here and $m = 0$ (axisymmetric) modes. Axisymmetric modes have received more attention than have non-axisymmetric ones. In particular, results of some generality (applying without restriction to wavenumber) are known about the dispersion relations for axisymmetric waves (Leibovich 1979). Furthermore, both $m = 0$ and $|m| = 1$ modes are necessarily involved in perturbations leading (in the most drastic cases) to vortex breakdown (see Leibovich 1984 for a discussion), so a study of their interaction would be of particular interest.

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